Differentiation

$(cu)' = cu'$ (c constant)

$(u + v)' = u' + v'$

$(uv)' = u'v + uv'$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$ (Chain rule)

$(x^n)' = nx^{n-1}$

$(e^x)' = e^x$

$(e^{ax})' = ae^{ax}$

$(a^x)' = a^x \ln a$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$

$(\sinh x)' = \cosh x$

$(\cosh x)' = \sinh x$

$(\ln x)' = \frac{1}{x}$

$(\log_a x)' = \frac{\log_a e}{x}$

$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}}$

$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$

$(\arctan x)' = \frac{1}{1 + x^2}$

$(\text{arccot}\, x)' = -\frac{1}{1 + x^2}$

Cauchy–Riemann equations[4]

$$u_x = v_y, \qquad u_y = -v_x$$

$$u_r = \frac{1}{r} v_\theta, \qquad v_r = -\frac{1}{r} u_\theta$$

$$\ln z = \ln r + i\theta \qquad (r = |z| > 0, \quad \theta = \arg z).$$

Ln z (Ln with capital L) and is called the **principal value** of ln z. Thus

$$\text{Ln}\, z = \ln |z| + i \,\text{Arg}\, z \qquad (z \neq 0).$$

generalized triangle inequality

$$|z_1 + z_2 + \cdots + z_n| \leqq |z_1| + |z_2| + \cdots + |z_n|;$$

$\sqrt[n]{z}$, for $z \neq 0$, has the n distinct values

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n}\right)$$

For arithmetic operations with **complex numbers**

(1) $$z = x + iy = re^{i\theta} = r(\cos\theta + i \sin\theta),$$

$r = |z| = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and for their representation in the complex plane, see Secs. 13.1 and 13.2.

A complex function $f(z) = u(x, y) + iv(x, y)$ is **analytic** in a domain D if it has a **derivative** (Sec. 13.3)

(2) $$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

everywhere in D. Also, $f(z)$ is *analytic at a point* $z = z_0$ if it has a derivative in a neighborhood of z_0 (not merely at z_0 itself).

If $f(z)$ is analytic in D, then $u(x, y)$ and $v(x, y)$ satisfy the (very important!) **Cauchy–Riemann equations** (Sec. 13.4)

(3) $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

everywhere in D. Then u and v also satisfy **Laplace's equation**

(4) $$u_{xx} + u_{yy} = 0, \qquad v_{xx} + v_{yy} = 0$$

everywhere in D. If $u(x, y)$ and $v(x, y)$ are continuous and have *continuous* partial derivatives in D that satisfy (3) in D, then $f(z) = u(x, y) + iv(x, y)$ is analytic in D. See Sec. 13.4. (More on Laplace's equation and complex analysis follows in Chap. 18.)

The **complex exponential function** (Sec. 13.5)

(5) $$e^z = \exp z = e^x (\cos y + i \sin y)$$

reduces to e^x if $z = x$ ($y = 0$). It is periodic with $2\pi i$ and has the derivative e^z.

The **trigonometric functions** are (Sec. 13.6)

(6) $$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}) = \cos x \cosh y - i \sin x \sinh y$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin x \cosh y + i \cos x \sinh y$$

and, furthermore,

$$\tan z = (\sin z)/\cos z, \qquad \cot z = 1/\tan z, \quad \text{etc.}$$

The **hyperbolic functions** are (Sec. 13.6)

(7) $$\cosh z = \tfrac{1}{2}(e^z + e^{-z}) = \cos iz, \qquad \sinh z = \tfrac{1}{2}(e^z - e^{-z}) = -i \sin iz$$

etc. The functions (5)–(7) are **entire**, that is, analytic everywhere in the complex plane.

The **natural logarithm** is (Sec. 13.7)

(8) $$\ln z = \ln|z| + i \arg z = \ln|z| + i \,\text{Arg}\, z \pm 2n\pi i$$

where $z \neq 0$ and $n = 0, 1, \cdots$. Arg z is the **principal value** of arg z, that is, $-\pi < \text{Arg}\, z \leqq \pi$. We see that ln z is infinitely many-valued. Taking $n = 0$ gives the **principal value** Ln z of ln z; thus $\text{Ln}\, z = \ln|z| + i \,\text{Arg}\, z$.

General powers are defined by (Sec. 13.7)

(9) $$z^c = e^{c \ln z} \qquad (c \text{ complex}, z \neq 0).$$